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on

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(August 1, 1964 to January 31, 1965)

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by

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INTRODUCTION

The main objective of this research project is to perform the elastic and viscoelastic analysis of two-dimensional problems with star-shaped boundaries by the method of complex variables. Since a successful application of this method requires the mapping of a given region conformally onto a unit circle, the first phase of the research is devoted to the development of a simple method of deriving satisfactory mapping functions. The work covering the part of this phase of research, which was submitted as the first semi-annual report (Feb. 1, 1964 to July 31, 1964), subsequently resulted in the following publication:

> Rim, K., and Stafford, R. O., "Derivation of Mapping Functions for Star-Shaped Regions," NASA CR-192, NASA, Washington, D. C., March, 1965.

The above report illustrates the hitherto unrecognized role played by the Schwarz-Christoffel transformation in the derivation of relatively simple mapping functions for seemingly complicated regions.

During the period covered by this status report, research effort was concentrated on

(a) further study on simplification of mapping functions,
(b) development of simple methods for <u>interior-to-interior</u> mapping, and
(c) review of the formulations for elastic analysis; and direction of succeeding research.

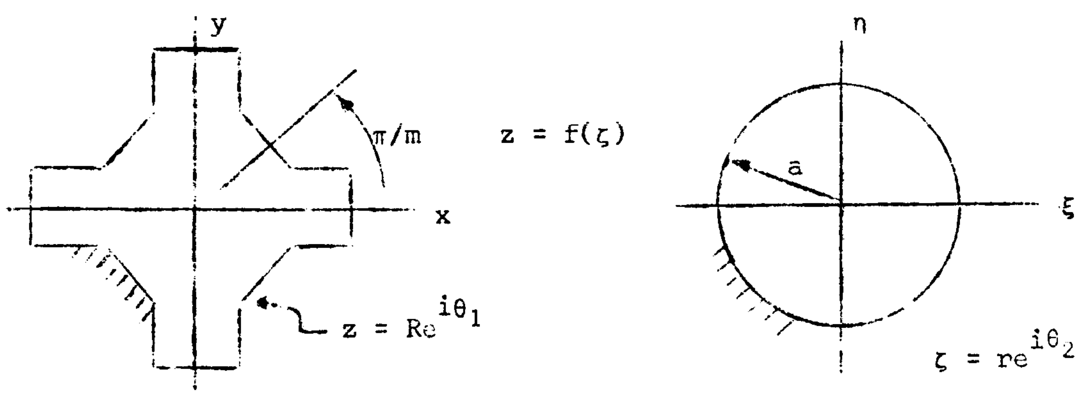
These three areas of effort will be explained in detail in the following text.

FURTHER STUDY ON SIMPLIFICATION OF MAPPING FUNCTIONS

Previous investigators (ref. 1 and 2) have developed several techniques for constructing approximate mapping functions. The authors (ref. 3) have also developed a rather simple method of constructing mapping functions for complicated symmetric polygons. However, the aforementioned methods have a common drawback in that their application to complex geometric figures requires extensive use of a high-speed digital computer and rather complicated programming.

To alleviate the requirements of complicated mathematical analysis and sophisticated computer programs, the authors have considered some new and simpler methods of constructing approximate mapping functions. It has been found that by assuming the mapping function to be a power series of a complex variable, and by imposing certain restrictions on the geometries, one can apply the theory of orthogonal functions and develop a completely general and very simple formula for determining the coefficients of the power series. A cursory development of the method will serve to illustrate its simplicity and generality.

Consider a general geometric figure in the z-plane which is to be mapped onto a circle (radius a) in the ζ-plane. To illustrate an important specific application and corresponding simplifications, a polygon with $2m$ axes of symmetry is shown.



z-plane   ζ-plane

In general, it could not be expected that θ_1 and θ_2 would be connected by any rational expression (ref. 2, pp. 501-505). The key to the development of approximate mapping functions is to assume a relationship between θ_1 and θ_2. Consider the simplest possible relation, that $\theta_1 = \theta_2$. This assumption requires that R be a single valued function of θ_1. Of course, more general relationship may be assumed, but only the simplest case is presented here for illustration.

Now take $z = f(\zeta)$ to be a power series in ζ with constant coefficients;

$$z = f(\zeta) = \sum_{k=0}^{\infty} A_k \zeta^{1-km}, \tag{1}$$

the form of the power series will determine the type of mapping; an exterior-to-exterior mapping function is shown. When $\zeta = ae^{i\theta}$, the power series must fit the polygon, i.e.,

$$\sum_{k=0}^{\infty} A_k a^{1-km} e^{i(1-k)m\theta} = R(\theta)e^{i\theta}$$

Multiplying this equation by $e^{-i(1-jm)\theta}$,

$$\sum_{k=0}^{\infty} A_k a^{1-km} e^{i(j-k)m\theta} = R(\theta)e^{i(jm)\theta},$$

and integrating it with respect to θ over one period, we obtain

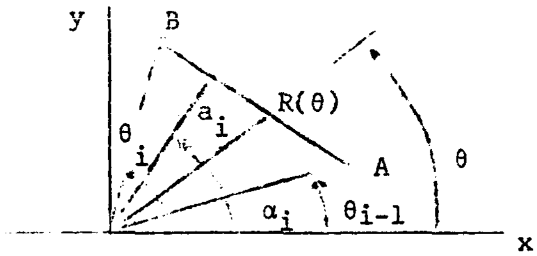
$$A_k = \frac{ma^{km-1}}{2\pi} \int_0^{\frac{2\pi}{m}} R(\theta)(\cos km\theta + i \sin km\theta)\, d\theta, \tag{2}$$

since

$$\int_0^{\frac{2\pi}{m}} e^{i(j-k)m\theta} d\theta = \begin{cases} 0, & j \neq k \\ \frac{2\pi}{m}, & j = k. \end{cases}$$

The integral in equation (2) may be readily evaluated as long as $R(\theta)$ satisfies the Dirichlet conditions over the specified interval.

Now consider the special case of a unit circle in the ζ-plane and a polygon with $2m$ axes of symmetry. The function $R(\theta)$ can arbitrarily be made an even function, thus A_k will be real. Since a polygon is composed of straight line segments, an analytic expression can be written for $R(\theta)$. Let there be n straight line segments between $\theta = 0$ and π/m ($n = 3$ is shown), and let points A and B denote the end points of the i-th line segment.



$$R(\theta) = \frac{a_i}{\cos(\alpha_i - \theta)}$$

$$\theta_{i-1} \leq \theta \leq \theta_i$$

Let a_i and α_i measure the length and orientation of a vector perpendicular to the i-th line segment, and let θ_{i-1} and θ_i be the angles of the end points of the line segment. Then by considering symmetry, equation (2) reduces to

$$A_k = \frac{m}{\pi} \sum_{i=1} a_i \int_{\theta_{i-1}}^{\theta_i} \frac{\cos km\theta}{\cos(\alpha_i - \theta)} d\theta \qquad (3)$$

Trigonometric integrals of the type shown in equations (2) and (3) cannot be accurately evaluated by standard quadrature formulas, but they can be easily computed to seven digit accuracy using a variation of Simpson's rule developed by L. N. G. Filon (ref. 4) in 1928. This method has been applied to several symmetric polygonal shapes with marked success. However, the restriction that $\theta_1 = \theta_2$ [$R(\theta_1)$ is single valued] limits the present method to a sub-group of the polygons that can be treated with the Schwarz-Christoffel transformation. This particular restriction may be removed by assuming more general relationship between θ_1 and θ_2.

The power and generality of the present method is seen by comparison to the other approximate methods of constructing mapping functions. In general, previous approximate methods involve some variation of the collocation technique, which requires an extensive computer programming effort (ref. 5) and often yields large polynomials.

The mapping functions derived by the present method can be readily calculated for any closed curvilinear shape, and reduce to the very simple equation (3) for polygonal shapes. Also, one can readily show that the coefficients A_k, equation (2), will give the best fit to the boundary in the sense of minimizing the square of the error. Thus the generality and simplicity of this method recommend it over the other approximate methods. The details of this method will be presented in the report which is being prepared for publication.

INTERIOR-TO-INTERIOR MAPPING

Simple methods for mapping the interior of a unit circle onto the interior of a given region are developed in a similar manner as in the authors' previous work (ref. 3 and the preceding derivations). For the interior-to-interior mapping, the converse of Remark II (given on p. 12, ref. 3, for the exterior-to-exterior mapping) holds true, and Figures 8 through 17 given in ref. 3 exhibit completely different trends.

These mapping functions are needed in the stress analysis of such common mechanical components as gears and splined shafts. Accurate determination of stress concentration factors is prerequisite to minimum weight design; and accurate stress analysis of such components requires precise mapping functions. Hence these mapping functions will enhance the capability of analyzing these and other vital machine elements. As an example, a splined shaft with sharp reentrant corners was analyzed and a stress concentration factor about 4 was obtained: This is higher than any previously reported analytic solution. It should be noted that such a problem would not only defy solution by known numerical methods, but would also present great experimental difficulties. These results will also be presented in the report being prepared for publication.

RESEARCH FOR THE SUCCEEDING HALF YEAR

Since the analysis of two-dimensional elasticity problems by the method of complex variables has been reasonably well formulated and the required mapping functions are now available, the investigators will concentrate on the viscoelastic analysis of star-shaped regions with time-dependent boundaries. The investigators will closely follow the steps outlined in the original proposal to NASA.

At the same time, a general numerical method of solving biharmonic boundary-value problems will be developed. This method does not require the use of mapping functions and would provide independent solutions which might be used to verify other analytical results. As described in the other proposal, it involves the numerical solution of integral equations resulting from the potential theory.

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